STABILITY OF THE SURFACE AREA PRESERVING MEAN CURVATURE FLOW IN EUCLIDEAN SPACE

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ABSTRACT. We show that the surface area preserving mean curvature flow in Euclidean space exists for all time and converges exponentially to a round sphere, if initially the L^2-norm of the traceless second fundamental form is small (but the initial hypersurface is not necessarily convex).

1. INTRODUCTION

Let M^n be a smooth, embedded, closed (compact, no boundary) n-dimensional manifold in $\mathbb{R}^{n+1}$, and we evolve it by the surface area preserving mean curvature flow, that is,

$$\frac{\partial F}{\partial t} = (1 - hH)\,\nu, \qquad F(\cdot, 0) = F_0(\cdot)\,. \tag{1.1}$$

Here $F_0 : M^n \to \mathbb{R}^{n+1}$ is the initial embedding, and $H = H(x,t)$ is the mean curvature and $\nu = \nu(x,t)$ is the outward unit normal vector of $M_t = F(\cdot,t)$ at point (x,t) (for simplicity, we simply write $(x,t) \in M_t$). And the function h is given by

$$h = h(t) = \frac{\int_{M_t} H\,d\mu}{\int_{M_t} H^2\,d\mu}\,, \tag{1.2}$$

where $d\mu = d\mu_t$ denotes the surface area element of the evolving surface M_t with respect to the induced metric $g(t)$. Clearly we have $H \not\equiv 0$ on M_0 since there is no closed minimal hypersurface in Euclidean space. A good monotonicity property of the surface area preserving mean curvature flow (1.1) is that the surface area of M_t remains unchanged and the volume of the $(n+1)$-dimensional region enclosed by M_t is non-decreasing along the flow, see Corollary 2.3.

We denote $A = \{a_{ij}\}$ as the second fundamental form of M_t and its traceless part as $\mathring{A} = A - \frac{H}{n}g$. Then we have $|\mathring{A}|^2 = |A|^2 - \frac{1}{n}H^2$. This quantity measures the roundness of the hypersurface.

In this paper, we prove the following theorem on the stability of this surface area preserving mean curvature flow:

Theorem 1.1. *Let $M_t^n \subset \mathbb{R}^{n+1}, n \geq 2$, be a smooth compact solution to the surface area preserving mean curvature flow (1.1) for $t \in [0,T)$ with $T \leq \infty$. Assume that*

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$h(0) > 0$. There exists $\epsilon > 0$, depending only on M_0 and $h(0)$, such that if

$$\int_{M_0} |\mathring{A}|^2 \, d\mu \le \epsilon \,, \tag{1.3}$$

then $T = \infty$ and the flow converges exponentially to a round sphere.

Remark 1.2. The idea of using an initial second fundamental form condition to pursue convergence of the flow was probably first studied in the case of Ricci flow ([**Ye93**]), later in the case of Kähler-Ricci flow ([**Che06, CLW09**] and others). The setting for the stability of the volume preserving mean curvature flow was studied by Escher-Simonett ([**ES98**]) and Li ([**Li09**]), under different set of conditions. Escher-Simonett's approach is a center manifold analysis while Li's approach is to apply a parabolic version of the Moser iteration method. Our approach is similar to the idea of iteration in [**Ye93, Li09**], in the cases of Ricci flow and volume preserving mean curvature flow, respectively. However, the analytical nature of our case, namely the surface area preserving mean curvature flow, is much more complicated than that of the volume preserving mean curvature flow, since the function $h(t)$ contains two integral terms both involving the mean curvature. Our approach is expected to be able to use to investigate the more general mixed volume mean curvature flow studied in ([**McC04**]).

Remark 1.3. In [**McC03**], McCoy proved that the surface area preserving mean curvature flow exists for all time and converges to a sphere if the initial hypersurface is *strictly convex*. As in the case of volume preserving mean curvature flow initiated by Huisken in [**Hui87**], strict convexity of the initial surface is essential. In our setting, we do not assume any convexity for the initial hypersurface. Under the conditions of the Theorem 1.1, evolving surfaces become *mean convex* instantly after flow starts (see equation (3.25)).

Outline of the proof: Our strategy is conventional: based on the initial bounds, we prove bounds on some time interval for several geometric quantities (Theorem 3.2), then we prove exponential decay for these quantities on the time interval of the interest (Theorem 3.6), which allows us to obtain uniform bounds for these quantities on the interval (Theorem 4.1), therefore we can repeat above arguments to extend the time interval (Theorem 4.2), and the amount of extension only depends on the initial conditions. Main theorem then follows.

Plan of the paper: There are four sections. In §2, we collect evolution equations for various geometric quantities associated to this flow, and provide some classic results that will be used in the proof. The proof of the main theorem is contained in the last two sections: we provide key estimates and prove exponential decay for $|\mathring{A}|$ and other quantities in §3, and we use these estimates to prove the long-time existence and exponential convergence in §4.

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2. PRELIMINARIES

We collect some necessary preliminary results in this section. In §2.1, we obtain evolution equations for some key quantities and operators, many of which were derived in [**McC03**]; in §2.2, we state and use Hamilton's interpolation inequalities for tensors to obtain a L^2 estimate (Lemma 2.11) on the gradients of the tensor $\mathring{A}$. A version of the parabolic maximum principle is also stated.

2.1. **Evolution of geometric quantities.** We start with the short time existence of the surface area preserving mean curvature flow (1.1) that is guaranteed by a work of Pihan:

Theorem 2.1. *([**Pih98**]) Let M_0 be a smooth embedded compact n-dimensional manifold in $\mathbb{R}^{n+1}$. Assume that $H \neq 0$ at some point of M_0 and $h(0) > 0$, then there exists $T_0 > 0$ such that the surface area preserving mean curvature flow (1.1) exists and is smooth for $t \in [0, T_0)$.*

We now collect and derive some evolution equations of several geometric quantities which will be used later. These quantities are:

(1) the induced metric of the evolving surface M_t: $g(t) = \{g_{ij}(t)\}$;
(2) the second fundamental form of M_t: $A(\bullet, t) = \{a_{ij}(\bullet, t)\}$, and its square norm given by
$$|A(\bullet, t)|^2 = g^{ij} g^{kl} a_{ik} a_{jl};$$
(3) the mean curvature of M_t with respect to the outward normal vector: $H(\bullet, t) = g^{ij} a_{ij}$;
(4) the traceless part of the second fundamental form: $\mathring{A} = A - \frac{H}{n} g$;
(5) the surface area element of M_t: $d\mu_t = \sqrt{det(g_{ij})}$.

Lemma 2.2. *([**McC03**]) The metric of M_t satisfies the evolution equation*
$$\frac{\partial}{\partial t} g_{ij} = 2(1 - hH) a_{ij}\,. \tag{2.1}$$
Therefore,
$$\frac{\partial}{\partial t} g^{ij} = -2(1 - hH) a^{ij} \tag{2.2}$$
and
$$\frac{\partial}{\partial t}(d\mu_t) = H(1 - hH) d\mu_t. \tag{2.3}$$
Moreover, the outward unit normal ν to M_t satisfies
$$\frac{\partial \nu}{\partial t} = h \nabla H\,. \tag{2.4}$$

As an easy consequence of (2.3), we have

Corollary 2.3. *([**McC03**])*

(1) *The surface area $|M_t|$ of M_t remains unchanged along the flow, i.e.,*
$$\frac{d}{dt} \int_{M_t} d\mu = \int_{M_t} (1 - hH) H \, d\mu = 0\,.$$

(2) *The volume of E_t, the $(n+1)$-dimensional region enclosed by M_t, is non-decreasing along the flow, i.e.,*

$$\frac{d}{dt}Vol(E_t) = \int_{M_t} d\mu - \frac{\left(\int_{M_t} H\, d\mu\right)^2}{\int_{M_t} H^2\, d\mu} \geq 0\,.$$

Remark 2.4. In Euclidean space, among all closed hypersurfaces, the sphere is of the least surface area with fixed enclosed volume, and as well as of the largest enclosed volume with fixed surface area. Therefore from this point of view, it is natural to study the sphere via both the volume preserving mean curvature flow and the surface area preserving mean curvature flow.

Theorem 2.5. *([McC03]) The second fundamental form satisfies the following evolution equation:*

$$\frac{\partial}{\partial t}a_{ij} = h\Delta a_{ij} + (1-2hH)a_i^{\,m}a_{mj} + h|A|^2 a_{ij}\,, \tag{2.5}$$

where $a_i^{\,m} = g^{ml}a_{li}$.

Corollary 2.6. *([McC03]) We have the evolution equations for H, $|A|^2$ and $|\mathring{A}|^2$:*

(i) $\frac{\partial}{\partial t}H = h\Delta H - (1-hH)|A|^2$;
(ii) $\frac{\partial}{\partial t}|A|^2 = h\left(\Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4\right) - 2tr\left(A^3\right)$,

where $tr\left(A^3\right) = g^{ij}g^{kl}g^{mn}a_{ik}a_{lm}a_{nj}$. Therefore we also have

(iii) $\frac{\partial}{\partial t}|\mathring{A}|^2 = h\Delta|\mathring{A}|^2 - 2h|\nabla\mathring{A}|^2 + 2h|A|^2|\mathring{A}|^2 - 2\left(tr(\mathring{A}^3) + \frac{2}{n}H|\mathring{A}|^2\right)$, *where* $|\nabla\mathring{A}|^2 = |\nabla A|^2 - \frac{1}{n}|\nabla H|^2$.

Proof. The last equation here is equivalent to the one from [McC03]. To see this, we used the following fact (see page 335 of [Li09]):

$$\mathrm{tr}\left(A^3\right) - \frac{1}{n}|A|^2 H = \mathrm{tr}\left(\mathring{A}^3\right) + \frac{2}{n}|\mathring{A}|^2 H.$$

□

We can then derive the evolution equations for the square norm of the gradients of the second fundamental form.

Corollary 2.7. *We have the evolution euqation for $|\nabla^m A|^2$:*

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^m A|^2 =& h\Delta|\nabla^m A|^2 - 2h|\nabla^{m+1}A|^2 + \sum_{i+j+k=m}\nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A\\ &+ \sum_{r+s=m}\nabla^r A * \nabla^s A * \nabla^m A\,,\end{aligned} \tag{2.6}$$

where $S * \Omega$ denotes any linear combination (involving h) of tensors formed by contraction on S and Ω by the metric g.

Proof. The time derivative of the Christoffel symbols Γ^i_{jk} is equal to

$$\frac{\partial}{\partial t}\Gamma^i_{jk} = \frac{1}{2}g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t}g_{kl}\right) + \nabla_k\left(\frac{\partial}{\partial t}g_{jl}\right) - \nabla_l\left(\frac{\partial}{\partial t}g_{jk}\right)\right\}$$

$$
\begin{aligned}
&= g^{il}\left\{\nabla_j\left((1-hH)a_{kl}\right)+\nabla_k\left((1-hH)a_{jl}\right)-\nabla_l\left((1-hH)a_{jk}\right)\right\}\\
&= A *_h \nabla A + \nabla A\,,
\end{aligned}
$$

where $*_h = *$ denotes the contraction on tensors involving h in the coefficients. Here we have also used the evolution equation for the metric, i.e., (2.1). Then we can proceed as in [**Ham82**, §13] (see also [**Hui84**, §7]) to get (2.6). □

In addition, we will need the following lemma on the time-derivative of the function $h(t) = \frac{\int_{M_t} H\,d\mu}{\int_{M_t} H^2 d\mu}$:

Lemma 2.8.

$$
\frac{\partial}{\partial t}h = \frac{\int_{M_t}[-(1-2hH)(1-hH)|A|^2 + H^2(1-hH)^2 + 2h^2|\nabla H|^2]d\mu}{\int_{M_t} H^2\,d\mu}.
$$

Proof. For the sake of completeness, we compute as follows:

$$
\begin{aligned}
&\frac{\partial}{\partial t}h = \frac{\partial}{\partial t}\left(\frac{\int_{M_t} H\,d\mu}{\int_{M_t} H^2\,d\mu}\right)\\
&= \left(\int_{M_t} H^2\,d\mu\right)^{-1}\left[\int_{M_t} -(1-hH)|A|^2 + H^2(1-hH)\,d\mu\right]\\
&\quad - \left(\int_{M_t} H^2\,d\mu\right)^{-1}\left[\int_{M_t} -2h^2|\nabla H|^2 - 2hH(1-hH)|A|^2 + hH^3(1-hH)\,d\mu\right]\\
&= \frac{\int_{M_t}[-(1-2hH)(1-hH)|A|^2 + H^2(1-hH)^2 + 2h^2|\nabla H|^2]d\mu}{\int_{M_t} H^2\,d\mu}\,.
\end{aligned}
$$

□

2.2. **Interpolation inequalities and maximum principle.** We will need the following Hamilton's interpolation inequalities for tensors.

Theorem 2.9. *([**Ham82**]) Let M be an n-dimensional compact Riemannian manifold and Ω be any tensor on M. Suppose*

$$
\frac{1}{p}+\frac{1}{q}=\frac{1}{r} \quad \textit{with } r\ge 1\,.
$$

We have the estimate

$$
\left(\int_M |\nabla\Omega|^{2r}\,d\mu\right)^{1/r} \le (2r-2+n)\left(\int_M |\nabla^2\Omega|^p\,d\mu\right)^{1/p}\left(\int_M |\Omega|^q\,d\mu\right)^{1/q}\,.
$$

Theorem 2.10. *([**Ham82**]) Let M and Ω be the same as the Theorem 2.9. If $1\le i\le n-1$ and $m\ge 0$, then there exists a constant $C=C(n,m)$ which is independent of the metric and connection on M, such that the following estimate holds:*

$$
\int_M |\nabla^i\Omega|^{2m/i}\,d\mu \le C\max_M |\Omega|^{2(m/i-1)}\int_M |\nabla^m\Omega|^2\,d\mu\,.
$$

As an application of these inequalities, we provide an estimate that will be used later.

Lemma 2.11. *For any $m \geq 1$ we have the estimate*

$$\frac{d}{dt}\int_{M_t} |\nabla^m A|^2 \, d\mu + 2h \int_{M_t} |\nabla^{m+1} A|^2 \, d\mu \leq C \max_{M_t} \left(|A|^2 + |A|\right) \int_{M_t} |\nabla^m A|^2 \, d\mu \,,$$

where $C = C(n, m, |h|)$.

Proof. By integrating (2.6) of Corollary 2.7 and using the generalized Hölder inequality we have

$$\begin{aligned}
&\frac{d}{dt}\int_{M_t} |\nabla^m A|^2 \, d\mu - \int_{M_t} (1 - hH)H|\nabla^m A|^2 \, d\mu + 2h \int_{M_t} |\nabla^{m+1} A|^2 \, d\mu \\
\leq & C \left\{ \left(\int_{M_t} |\nabla^i A|^{2m/i} \, d\mu\right)^{i/2m} \left(\int_{M_t} |\nabla^j A|^{2m/j} \, d\mu\right)^{j/2m} \left(\int_{M_t} |\nabla^k A|^{2m/k} \, d\mu\right)^{k/2m} \right. \\
& \left. + \left(\int_{M_t} |\nabla^r A|^{2m/r} \, d\mu\right)^{r/2m} \left(\int_{M_t} |\nabla^s A|^{2m/s} \, d\mu\right)^{s/2m} \right\} \left(\int_{M_t} |\nabla^m A|^2 \, d\mu\right)^{1/2} ,
\end{aligned}$$

with $i + j + k = r + s = m$.

Applying Lemma 2.10 for tensor A, we get

$$\left(\int_{M_t} |\nabla^q A|^{2m/q} \, d\mu\right)^{q/2m} \leq C \max_{M_t} |A|^{1-q/m} \left(\int_{M_t} |\nabla^m A|^2 \, d\mu\right)^{1/2m} ,$$

where $q = i, j, k, r, s$.

Also note that

$$\begin{aligned}
\int_{M_t} |(1 - hH)H|\nabla^m A|^2 d\mu &\leq \max_{M_t}\{|H| + |h|H^2\} \int_{M_t} |\nabla^m A|^2 d\mu \\
&\leq C(n, |h|) \max_{M_t} \left(|A|^2 + |A|\right) \int_{M_t} |\nabla^m A|^2 d\mu.
\end{aligned}$$

Combining these inequalities thus completes the proof. □

We will need the following version of the maximum principle, especially in the proof of the Theorem 3.2.

Theorem 2.12. *(Maximum principle, see e.g.* [**CLN06**, Lemma 2.12]*) Suppose $u : M \times [0, T] \to \mathbb{R}$ satisfies*

$$\frac{\partial}{\partial t} u \leq a^{ij}(t)\nabla_i \nabla_j u + \langle B(t), \nabla u\rangle + F(u) \,,$$

where the coefficient matrix $\left(a^{ij}(t)\right) > 0$ for all $t \in [0, T]$, $B(t)$ is a time-dependent vector field and F is a Lipschitz function. If $u \leq c$ at $t = 0$ for some $c > 0$, then $u(x, t) \leq U(t)$ for all $(x, t) \in M_t, t \geq 0$, where $U(t)$ is the solution to the following initial value problem:

$$\frac{d}{dt} U(t) = F(U) \quad \textit{with} \quad U(0) = c \,.$$

3. Proof of Theorem 1.1: estimates

We break our proof into two sections. In this section, we provide key estimates that will be needed: in §3.1, we establish the L^∞-bound for $\mathring{A}$ from its L^2 bound; in §3.2, we prove the exponential decay for $|\mathring{A}|$.

3.1. Establishing bounds for geometric quantities.

Let us start with a result of Topping which plays an important role in the key estimates we will focus on in this subsection.

Lemma 3.1. *([**Top08**]) Let M be an n-dimensional closed, connected manifold smoothly immersed in $\mathbb{R}^N$, where $N \geq n+1$. Then the intrinsic diameter and the mean curvature H of M are related by*

$$diam\,(M) \leq C(n) \int_M |H|^{n-1}\, d\mu\,.$$

We now prove the following key estimates. This allows us to obtain the L^∞-bound for $\mathring{A}$, $|\nabla H|$ and $|1-hH|$ on some time interval. More specifically,

Theorem 3.2. *Let $M_t^n \subset \mathbb{R}^{n+1}, n \geq 2$, be a smooth compact solution to the surface area preserving mean curvature flow (1.1) for $t \in [0,T)$ with $T \leq \infty$. Assume that*

$$\max_{M_0} |A| \leq \Lambda_0 \quad \text{and} \quad \frac{1}{\Lambda_0} \leq \left\{ h(0)\,, \int_{M_0} H^2\, d\mu\,, \int_{M_0} |\nabla^m A|^2\, d\mu \right\} \leq \Lambda_0, \tag{3.1}$$

for some $\Lambda_0 \geq 2$ sufficiently large and all $m \in [1, \widehat{m}]$ with $\widehat{m}$ sufficiently large. Then there exists $\epsilon_0 = \epsilon_0(n, |M_0|, \Lambda_0) > 0$ and $T_1 = T_1(\Lambda_0) \leq 1$, such that if

$$\int_{M_0} |\mathring{A}|^2\, d\mu \leq \epsilon \leq \epsilon_0\,, \tag{3.2}$$

then for all $t \in [0, T_1]$ we have

$$\max_{M_t} |A| \leq 2\Lambda_0 \quad \text{and} \quad \frac{1}{2\Lambda_0} \leq \left\{ h(t)\,, \int_{M_t} H^2\, d\mu \right\} \leq 2\Lambda_0\,. \tag{3.3}$$

Moreover, there exists $C_1 = C_1(n, |M_0|, \Lambda_0)$ and some universal constant $\alpha \in (0,1)$ such that for any $t \in [0, T_1]$

$$\max_{M_t} \big(|\mathring{A}| + |\nabla H| + |1-hH|\big) \leq C_1 \epsilon^\alpha\,. \tag{3.4}$$

Remark 3.3. It is very important to keep track of the dependence of constants on geometric quantities. As we shall see from the proof below, the constant $C_1 = C_1(n, |M_0|, \Lambda_0)$ is non-decreasing in Λ_0.

Proof. By the short time continuity, we first let $t_1 > 0$ be the maximal time such that for all $t \in [0, t_1]$ we have

$$\max_{M_t} |A| \leq 2\Lambda_0 \quad \text{and} \quad \frac{1}{2\Lambda_0} \leq \left\{ h(t)\,, \int_{M_t} H^2\, d\mu \right\} \leq 2\Lambda_0\,. \tag{3.5}$$

Now using the fact that $|\mathrm{tr}\,(A^3)\,| \leq |A|^3$ (see Lemma 2.2 [**HS99**]), and Kato's inequality $|\nabla |A|| \leq |\nabla A|$, we derive from (ii) of Corollary 2.6 to find

$$\frac{\partial}{\partial t}|A| \leq h\Delta |A| + 2\Lambda_0 |A|^3 + |A|^2 \quad \text{on } M_t \text{ for all } t \in [0, t_1]\,.$$

Then by the maximum principle (Theorem 2.12), we have:

$$\max_{M_t} |A| \leq U(t) \text{ for all } t \in [0, t_1]\,, \text{ with } U(0) = \Lambda_0,$$

where $U(t) > 0$ solves

$$2\Lambda_0 \ln\left(2\Lambda_0 + \frac{1}{U}\right) - \frac{1}{U} = t + 2\Lambda_0 \ln\left(2\Lambda_0 + \frac{1}{\Lambda_0}\right) - \frac{1}{\Lambda_0}\,.$$

Therefore, there exists $0 < t_2 = t_2(\Lambda_0) \leq 1$ such that

$$\max_{M_t} |A| \leq \frac{3\Lambda_0}{2} \quad \text{for all } t \in [0, t_2]\,. \tag{3.6}$$

Then the first assertion of the Theorem, namely, (3.3), is obtained from the following technical lemma by setting $T_1 = \min\{t_1, t_2\}$.

Lemma 3.4. *There exists some constant $\epsilon_0 = \epsilon_0(n, |M_0|, \Lambda_0) > 0$ such that if the condition (3.2) is satisfied, then*

$$t_1 \geq t_2 = t_2(\Lambda_0)\,.$$

Proof. of the Lemma 3.4: Suppose this is not the case, then we have $t_1 < t_2 \leq 1$. Then by (3.5) and (3.6), we deduce that at time $t = t_1$ either $h(t)$ or $\int_{M_t} H^2 d\mu$ achieves the extreme value $2\Lambda_0$ or $\frac{1}{2\Lambda_0}$.

Now since $\{\max_{M_t} |A|, h(t)\} \leq 2\Lambda_0$ for all $t \in [0, t_1]$, integrating the equation (2.6) of the Corollary 2.7 over M_t, and using Hamilton's interpolation inequality for tensors (Lemma 2.10), we have the uniform bound on all the higher order derivatives of A, which only depends on n and Λ_0 (more precisely, $\max_t |h(t)|, \max_{M_t} |A|$ and the initial bound on the L^2-norm of all the derivatives of A in (3.1)). In particular, for all $m \in [1, \widehat{m}]$, we have:

$$\max_{M_t} |\nabla^m A| \leq C(n, \Lambda_0) \quad \text{for } \; t \in [0, t_1]\,, \tag{3.7}$$

c.f. [**Hui84**, Lemma 8.3].

Now we integrate the evolution equation for $\mathring{A}$, namely, the equation (iii) of the Corollary 2.6 over M_t for $t \in [0, t_1]$, to get

$$\begin{aligned}
&\frac{\partial}{\partial t} \int_{M_t} |\mathring{A}|^2 \, d\mu - \int_{M_t} |\mathring{A}|^2 H(1 - hH) \, d\mu \\
&= \int_{M_t} \left[-2h|\nabla \mathring{A}|^2 + 2h|A|^2|\mathring{A}|^2 - 2\left(\operatorname{tr}(\mathring{A}^3) + \frac{2}{n} H |\mathring{A}|^2\right)\right] d\mu\,,
\end{aligned}$$

and therefore

$$\frac{\partial}{\partial t} \int_{M_t} |\mathring{A}|^2 \, d\mu \leq C(n, \Lambda_0) \int_{M_t} |\mathring{A}|^2 \, d\mu \quad \text{for all } t \in [0, t_1]\,, \tag{3.8}$$

where we have used $|H| \leq \sqrt{n}|A| \leq 2\sqrt{n}\Lambda_0$ and $|\operatorname{tr}(\mathring{A}^3)| \leq |\mathring{A}|^3 \leq 2\Lambda_0 |\mathring{A}|^2$.

Therefore, using (3.8) and the assumption that $\int_{M_0} |\mathring{A}|^2 \, d\mu \leq \epsilon \leq \epsilon_0$, we now get

$$\int_{M_t} |\mathring{A}|^2 \, d\mu \leq \epsilon e^{C(n, \Lambda_0) t} \leq C(n, \Lambda_0)\epsilon \quad \text{for all } t \in [0, t_1]\,, \tag{3.9}$$

where we abuse our notation for $C(n, \Lambda_0)$.

Now we observe from Hamilton's interpolation inequality (Lemma 2.9 with $r = 1, p = q = 2$):

$$\int_{M_t} |\nabla \mathring{A}|^2 \, d\mu \leq n \left(\int_{M_t} |\mathring{A}|^2 \, d\mu \right)^{\frac{1}{2}} \left(\int_{M_t} |\nabla^2 \mathring{A}|^2 \, d\mu \right)^{\frac{1}{2}} \leq C(n, \Lambda_0) \epsilon^{\frac{1}{2}} \,, \tag{3.10}$$

where we used $|\nabla^2 \mathring{A}| \leq C(n) |\nabla^2 A|$ and (3.7). In fact, using (3.7) and applying Lemma 2.9 inductively, we have, for all $m \in [1, \widehat{m}]$,

$$\int_{M_t} |\nabla^m \mathring{A}|^2 \, d\mu \leq C(n, m, \Lambda_0) \epsilon^{\frac{1}{2m}} \quad \text{for all } t \in [0, t_1] \,. \tag{3.11}$$

This together with Lemma 2.10 imply that, for all $t \in [0, t_1]$,

$$\int_{M_t} |\nabla^m \mathring{A}|^p \, d\mu \leq C(n, m, p, \Lambda_0) \epsilon^{\frac{1}{mp}} \quad \text{for all } m \in [1, \widehat{m}] \text{ and } p < \infty.$$

This yields, by the standard Sobolev inequality (see e.g. [**Aub98**, §2]), that for some universal constant $\alpha \in (0, 1)$, and all $m \in [1, \widehat{m}]$, and $t \in [0, t_1]$, we have

$$\max_{M_t} |\nabla^m \mathring{A}| \leq C(n, m, \Lambda_0) \epsilon^{\alpha}. \tag{3.12}$$

In particular, using [**Hui84**, Lemma 2.2], for all $t \in [0, t_1]$ we have

$$\max_{M_t} |\nabla H| \leq C(n) \max_{M_t} |\nabla \mathring{A}| \leq C_1(n, \Lambda_0) \epsilon^{\alpha} \,. \tag{3.13}$$

Therefore, using (3.5) (3.13) and Topping's Theorem 3.1, we have

$$\begin{aligned} |1 - hH|(x, t) &= \left(\int_{M_t} H^2 \, d\mu \right)^{-1} \left| \int_{M_t} H^2 \, d\mu - H(x, t) \int_{M_t} H \, d\mu \right| \\ &\leq 2 \Lambda_0^{-1} \mathrm{diam}\,(M_t) \int_{M_t} |\nabla H| |H| \, d\mu \\ &\leq C_1(n, |M_0|, \Lambda_0) \epsilon^{\alpha} \,, \end{aligned} \tag{3.14}$$

for all $(x, t) \in M_t$ and all $t \in [0, t_1]$. Here we abuse the notation on C_1 but we are allowed to choose a larger constant C_1 than previously in (3.13).

Now consider the evolution equation for H^2, that is (see the equation (i) of Corollary 2.6)

$$\frac{\partial}{\partial t} H^2 = h \Delta H^2 - 2h |\nabla H|^2 - 2(1 - hH) H |A|^2 \,. \tag{3.15}$$

Integrating this over M_t we get

$$\frac{\partial}{\partial t} \int_{M_t} H^2 \, d\mu = \int_{M_t} H^3 (1 - hH) - 2h |\nabla H|^2 - 2(1 - hH) H |A|^2 \, d\mu \,, \tag{3.16}$$

where the first term on the right-hand side comes from the time derivative of μ_t, i.e., the equation (2.3).

Therefore we have (using $0 < \epsilon \leq 1$):

$$\left| \frac{\partial}{\partial t} \int_{M_t} H^2 \, d\mu \right| \leq C(n, \Lambda_0) |M_0| \epsilon^{\alpha} \quad \text{for all } t \in [0, t_1] \,. \tag{3.17}$$

Similarly, using the evolution equation for h, that is, the Lemma 2.8, we have

$$\left|\frac{\partial}{\partial t}h\right| \leq C(n,\Lambda_0)|M_0|\epsilon^\alpha \quad \text{for all } t\in[0,t_1]\,. \tag{3.18}$$

Integrating (3.17) and (3.18) over $[0,t_1]$ (note that $t_1 < t_2 \leq 1$), and choosing $\epsilon_0 = \epsilon_0(n,|M_0|,\Lambda_0) \geq \epsilon$ sufficiently small, we obtain

$$\frac{2}{3\Lambda_0} \leq \left\{h(t_1)\,,\ \int_{M_{t_1}} H^2\,d\mu\right\} \leq \frac{3\Lambda_0}{2}\,.$$

This contradicts with the assumption that either $h(t_1)$ or $\int_{M_{t_1}} H^2 d\mu$ achieves the extreme value $2\Lambda_0$ or $\frac{1}{2\Lambda_0}$. Therefore $T_1 = t_2(\Lambda_0)$. □

To see (3.4), we can repeat the above argument by replacing t_1 by $t_2 = T_1$. Note that (3.9), (3.12) and Lemma 3.1 together yield a bound on $\max_{M_t}|\mathring{A}|$ in terms of $n, |M_0|, \Lambda_0$ and ϵ for all $t\in[0,t_1]$. Bounds for $|\nabla H|$ and $|1-hH|$ are as in (3.13) and (3.14), respectively. We will still call this bound $C_1 = C_1(n,|M_0|,\Lambda_0)$ which is chosen to be larger than the C_1's in (3.13) and (3.14). Now the proof is complete. □

Remark 3.5. Our conditions (3.1) appear to be necessary for the case of the volume preserving mean curvature flow in [**Li09**] for the initial hypersurface. Moreover, a more general form of the Kato's inequality for $|\nabla\mathring{A}|$ is probably not true, but appears to be necessary to deploy Moser's parabolic iteration in [Pages 337, 338 and 340, [**Li09**]]. We do not use the iteration method here to obtain the L^∞ bounds for $|\mathring{A}|$, $|\nabla H|$ and $|1-hH|$.

3.2. **Establishing the exponential decay for geometric quantities.** Previously we have obtained a time $T_1 = T_1(\Lambda_0)$ which only depends on the initial hypersurface, and $\epsilon_0 = \epsilon_0(n,|M_0|,\Lambda_0)$ small enough such that if the initial L^2 norm of $\mathring{A}$ is small (see (3.2)), then we have estimates (3.3) and (3.4) on time interval $[0,T_1]$. In this subsection, we show that if on some time interval $[0,T)$, estimates similar to (3.3) and (3.4) hold, then we can choose an ϵ small enough for the initial L^2 bound on $\mathring{A}$, such that $|\mathring{A}|$, $|\nabla H|$ and $|1-hH|$ decay exponentially on this time interval $[0,T)$. More precisely,

Theorem 3.6. *Let $M_t^n\subset\mathbb{R}^{n+1}, n\geq 2$, be a smooth compact solution to the surface area preserving mean curvature flow (1.1) with initial condition (3.1). Suppose that for any $t\in[0,T), T\leq\infty$, we have*

$$\max_{M_t}|A| \leq \Lambda_1 \quad \textit{and} \quad \frac{1}{\Lambda_1} \leq \left\{h(t)\,,\ \int_{M_t} H^2\,d\mu\right\} \leq \Lambda_1 \tag{3.19}$$

and

$$\max_{M_t}\left(|\mathring{A}| + |\nabla H| + |1-hH|\right) \leq \hat{C}\epsilon^\beta \tag{3.20}$$

for some $\beta > 0$. Then there exists $\epsilon_1 = \epsilon_1(n,|M_0|,\Lambda_1,\hat{C},\beta) > 0$ such that if

$$\int_{M_0}|\mathring{A}|^2\,d\mu \leq \epsilon \leq \epsilon_1\,, \tag{3.21}$$

then for all $t \in [0,T)$ we have

$$\max_{M_t} |\mathring{A}| \leq \left(\max_{M_0} |\mathring{A}|\right) e^{-\delta t} \tag{3.22}$$

and

$$\max_{M_t} \left(|\mathring{A}| + |\nabla H| + |1 - hH|\right) \leq C_2(n, |M_0|, \Lambda_1, \hat{C}) \left(\max_{M_0} |\mathring{A}|\right)^{\alpha} e^{-\alpha\delta t}, \tag{3.23}$$

where $\delta = \frac{1}{4n\Lambda_1^2|M_0|} > 0$, and $\alpha \in (0,1)$ is from Theorem 3.2.

Remark 3.7. To directly apply the results of Theorem 3.2, we may take $\beta = \alpha$, and $\Lambda_1 = 2\Lambda_0$, as well as $\hat{C} = C_1$. We state the Theorem 3.6 this way so that it can be easily adapted in later applications.

Proof. Using (3.19), for any $t \in [0,T)$ we have

$$\int_{M_t} H\, d\mu = h(t) \int_{M_t} H^2\, d\mu \geq \frac{1}{\Lambda_1^2}.$$

Therefore we can always find a point $x_0 \in M_t$ (which may depend on $t \in [0,T)$) such that

$$H(x_0,t) \geq \frac{1}{\Lambda_1^2 |M_t|} = \frac{1}{\Lambda_1^2 |M_0|}. \tag{3.24}$$

Now by (3.19), (3.20), the inequality $|H| \leq \sqrt{n}|A|$, and the Theorem 3.1, for fixed $t \in [0,T)$, we have

$$\begin{aligned} H(x,t) - H(x_0,t) &\geq -\max|\nabla H| \mathrm{diam}(M_t) \\ &\geq -\hat{C}\epsilon^{\beta} C(n) \int_{M_t} H^{n-1} d\mu \\ &\geq -\hat{C}\epsilon^{\beta} C(n) n^{\frac{n-1}{2}} |M_0| \Lambda_1^{n-1}. \end{aligned}$$

Now we can apply (3.24), and choose $\epsilon_1 = \epsilon_1(n, |M_0|, \Lambda_1, \hat{C}, \beta) > 0$ sufficiently small, such that if $\epsilon \leq \epsilon_1$, then for all $t \in [0,T)$, we have

$$\min_{M_t} H \geq H(x_0,t) - \hat{C}\epsilon^{\beta} C(n) n^{\frac{n-1}{2}} |M_0| \Lambda_1^{n-1} \geq \frac{1}{2\Lambda_1^2 |M_0|} > 0. \tag{3.25}$$

Moreover, we have

$$\begin{aligned} hH^2 &= \frac{H^2 \int_{M_t} H d\mu}{\int_{M_t} H^2 d\mu} \\ &\leq \frac{H^2 \int_{M_t} \{H(x_0,t) + \max|\nabla H| \mathrm{diam}(M_t)\} d\mu}{\int_{M_t} \{H(x_0,t) - \max|\nabla H| \mathrm{diam}(M_t)\}^2 d\mu} \\ &\leq \frac{\{H(x_0,t) + \max|\nabla H| \mathrm{diam}(M_t)\}^3}{\{H(x_0,t) - \max|\nabla H| \mathrm{diam}(M_t)\}^2}. \end{aligned}$$

We can then apply the estimate on $|\nabla H|$, namely (3.20), we can choose ϵ_1 small enough such that from above, we have

$$hH^2 \leq \frac{3}{2} H(x_0,t) - 2\max(|\nabla H|)\mathrm{diam}(M_t).$$

This implies, by choosing a possibly smaller ϵ_1, we have

$$\begin{aligned}\max_{M_t}\left(\frac{2}{n}hH^2-\frac{4}{n}H\right) &= \max_{M_t}\frac{2}{n}\left(hH^2-2H\right)\\ &\le \frac{2}{n}\left(\frac{3}{2}H(x_0,t)-2H(x_0,t)\right)\\ &\le -\frac{1}{n\Lambda_1^2|M_0|}\,. \end{aligned} \tag{3.26}$$

Here in the last step, we applied (3.24).

To derive exponential decay for $|\mathring{A}|^2$, we recall its evolution equation, namely, (iii) of Corollary 2.6, we have

$$\begin{aligned}\frac{\partial}{\partial t}|\mathring{A}|^2 &= h\Delta|\mathring{A}|^2-2h|\nabla\mathring{A}|^2+2h|A|^2|\mathring{A}|^2-2\left(\mathrm{tr}(\mathring{A}^3)+\frac{2}{n}H|\mathring{A}|^2\right)\\ &\le h\Delta|\mathring{A}|^2+\left(2h(|\mathring{A}|^2+\frac{1}{n}H^2)+2|\mathring{A}|-\frac{4}{n}H\right)|\mathring{A}|^2\\ &= h\Delta|\mathring{A}|^2+(2h|\mathring{A}|^2+2|\mathring{A}|)|\mathring{A}|^2+\left(\frac{2}{n}hH^2-\frac{4}{n}H\right)|\mathring{A}|^2\end{aligned}$$

Now we use the inequality $|\mathring{A}|\le \hat{C}\epsilon^\beta$ ((3.20)), and the inequality (3.26), by choosing a possibly smaller ϵ_1, we have for $\epsilon<\epsilon_1$,

$$\frac{\partial}{\partial t}|\mathring{A}|^2 \le h\Delta|\mathring{A}|^2-\frac{1}{2n\Lambda_1^2|M_0|}|\mathring{A}|^2 = h\Delta|\mathring{A}|^2-2\delta|\mathring{A}|^2, \tag{3.27}$$

where $\delta=\frac{1}{4n\Lambda_1^2|M_0|}$.

Therefore the exponential decay of $|\mathring{A}|$, namely the estimate (3.22) now follows from the maximum principle (Theorem 2.12).

Finally, once we obtain (3.22), we can prove (3.23) exactly following the argument in the proof of the Theorem 3.2 (see (3.10)–(3.14)). □

4. Proof of Theorem 1.1: continued

We now assemble estimates obtained from last section to complete the proof of the main theorem: in §4.1, we prove the long-time existence of the flow (1.1) by establishing the uniform upper bound for $|A|$; in §4.2, we show the exponential convergence of the flow.

4.1. Extending the time interval.

In the previous section, we obtain the exponential decay for $|\mathring{A}|$, $|\nabla H|$ and $|1-hH|$ on some time interval. We will next show that this implies a uniform bound on the function h, which consequently yields the uniform bound on $|A|$. We will state the theorem in a more general form in order for later application.

Theorem 4.1. *Let $M_t^n\subset\mathbb{R}^{n+1}, n\ge 2$, be a smooth compact solution to the surface area preserving mean curvature flow (1.1) with initial conditions (3.1) and (3.2).*

Suppose that for any $t \in [0, T), T \leq \infty$ *we have*

$$\max_{M_t} |A| \leq \Lambda_2 \quad \text{and} \quad \frac{1}{\Lambda_2} \leq \left\{ h(t)\,, \int_{M_t} H^2 \, d\mu \right\} \leq \Lambda_2 \tag{4.1}$$

and

$$\max_{M_t} \left(|\mathring{A}| + |\nabla H| + |1 - hH| \right) \leq \tilde{C} \epsilon^\beta e^{-\alpha \delta t}\,, \tag{4.2}$$

for some $\beta, \delta > 0$ *and* $\alpha > 0$*, where* α *is the same as in the Theorem 3.2, and* β, δ *are the same as in the Theorem 3.6.*

Then we have the following uniform estimate for all $t \in [0, T)$*:*

$$\int_{M_0} H^2 \, d\mu - b_0 \delta^{-1} \epsilon^\beta \leq \int_{M_t} H^2 \, d\mu \leq \int_{M_0} H^2 \, d\mu + b_0 \delta^{-1} \epsilon^\beta\,, \tag{4.3}$$

where $b_0 = b_0(n, |M_0|, \Lambda_2, \tilde{C}) = 2(n^{\frac{3}{2}} \Lambda_2^2 + \tilde{C}) \Lambda_2 |M_0| \tilde{C}$ *and*

$$h(0) - b_1 \delta^{-1} \epsilon^\beta \leq h(t) \leq h(0) + b_1 \delta^{-1} \epsilon^\beta\,, \tag{4.4}$$

where $b_1 = b_1(n, |M_0|, \Lambda_2, \tilde{C}) = (2n^{\frac{1}{2}} \Lambda_2^2 + 2n\tilde{C} + 1) \Lambda_2^3 |M_0| \tilde{C}$.

Moreover, there exists $\epsilon_2 = \epsilon_2(n, |M_0|, \Lambda_2, \tilde{C}, \beta, \delta) > 0$ *such that if* $\epsilon \leq \epsilon_2$ *then for any* $t \in [0, T)$

$$\max_{M_t} |A| \leq \Lambda_0^4, \tag{4.5}$$

and

$$\frac{1}{\Lambda_0^4} \leq \left\{ h(t)\,, \int_{M_t} H^2 \, d\mu\,, \int_{M_t} |\nabla^m A|^2 \, d\mu \right\} \leq \Lambda_0^4, \tag{4.6}$$

for all $m \in [1, \widehat{m}]$*, where* $\widehat{m}$ *and* Λ_0 *are from the Theorem 3.2.*

Proof. Without loss of generality, we assume $\epsilon \leq 1$. We start by recalling the following integral (3.16):

$$\frac{\partial}{\partial t} \int_{M_t} H^2 \, d\mu = \int_{M_t} H^3(1 - hH) - 2h|\nabla H|^2 - 2(1 - hH) H |A|^2 \, d\mu.$$

Using the assumptions (4.1) and (4.2), we can estimate this integral term by term as follows:

$$|H^3(1 - hH)| \leq n^{\frac{3}{2}} \Lambda_2^3 \tilde{C} \epsilon^\beta e^{-\delta t},$$

and

$$2h|\nabla H|^2 \leq 2\Lambda_2 \tilde{C}^2 \epsilon^{2\beta} e^{-2\delta t},$$

and

$$2H|1 - hH||A|^2 \leq 2n^{\frac{1}{2}} \Lambda_2^3 \tilde{C} \epsilon^\beta e^{-\delta t}.$$

Putting these estimates together, and abusing our notation for $\tilde{C}$ and δ, we obtain

$$\left| \frac{\partial}{\partial t} \int_{M_t} H^2 \, d\mu \right| \leq \tilde{C} \epsilon^\beta e^{-\delta t} \quad \text{for all } t \in [0, T). \tag{4.7}$$

Integrating this over $[0, t]$ for any $t \leq T$ we obtain the estimate (4.3).

In order to show the estimate (4.4), we use the evolution equation of h, namely, the Lemma 2.8:

$$\frac{\partial}{\partial t}h = \frac{\int_{M_t}[-(1-2hH)(1-hH)|A|^2 + H^2(1-hH)^2 + 2h^2|\nabla H|^2]d\mu}{\int_{M_t} H^2\, d\mu}.$$

We again estimate it term by term, under the assumptions (4.1), (4.2). Up to abuse of the notation for $\tilde{C} = \tilde{C}(n, |M_0|, \Lambda_2)$ and δ, assuming again that $\epsilon \le 1$, we have for any $t \in [0, T)$:

$$\left|\frac{\partial}{\partial t}h\right| \le \tilde{C}\epsilon^{\beta}e^{-\delta t}. \tag{4.8}$$

Integrating this over $[0, t]$ for any $t \le T$ we get (4.4).

We then use

$$|H| \le \frac{|1-hH|}{h} + \frac{1}{h},$$

and we choose $\epsilon_2 = \epsilon_2(n, |M_0|, \Lambda_2, \tilde{C}, \beta, \delta) > 0$ sufficiently small, in view of the initial condition (3.1) ($\frac{1}{\Lambda_0} \le h(0) \le \Lambda_0$), and (4.2)–(4.4), we get, for any $t \in [0, T)$,

$$\begin{aligned}\max_{M_t}|H| &\le \frac{|1-hH|}{h} + \frac{1}{h}\\ &\le \frac{5}{4h}\\ &\le \frac{5}{4(h(0) - \frac{1}{4\Lambda_0})}\\ &\le \frac{5\Lambda_0}{3}.\end{aligned}$$

and thus

$$\max_{M_t}|A| \le \max_{M_t}\left(\sqrt{|\mathring{A}|^2 + \frac{1}{n}|H|^2}\right) \le 2\Lambda_0 \le \Lambda_0^4, \tag{4.9}$$

where we can choose Λ_0 large enough for the last inequality. This proves (4.5).

The bound on $h(t)$ and $\int_{M_t} H^2\, d\mu$ in (4.6) follows immediately from (3.1), (4.3) and (4.4). We are left to estimate the integral $\int_{M_t}|\nabla^m A|^2\, d\mu$.

Now the Lemma 2.11 yields for all $m \in [1, \widehat{m}]$,

$$\begin{aligned}\frac{d}{dt}\int_{M_t}|\nabla^m A|^2\, d\mu &\le C(n, m, |h|)\max_{M_t}\{|A|^2 + |A|\}\int_{M_t}|\nabla^m A|^2\, d\mu\\ &\le C'(n, m, |h|)\Lambda_0^2\int_{M_t}|\nabla^m A|^2\, d\mu. \end{aligned}\tag{4.10}$$

Since we are allowed to assume Λ_0 in Theorem 3.2 is sufficiently large so that

$$\Lambda_0 \ge \max\{C'(n, m, |h|), 100\},$$

then we obtain from the inequality (4.10) that

$$\int_{M_t}|\nabla^m A|^2\, d\mu \le C'(n, m, |h|)\Lambda_0^2\int_{M_0}|\nabla^m A|^2\, d\mu \le \Lambda_0^4.$$

Now our proof is complete. ∎

We now complete the proof for long-time existence of the flow by the following extension theorem:

Theorem 4.2. *Let $M_t^n \subset \mathbb{R}^{n+1}, n \geq 2$, be a smooth compact solution to the surface area preserving mean curvature flow* (1.1) *with initial conditions* (3.1) *and* (3.2)*. Suppose that for any $t \in [0,T], T < \infty$ and for all $m \in [1, \widehat{m}]$ we have*

$$\max_{M_t} |A| \leq \Lambda_0^4 \quad \text{and} \quad \frac{1}{\Lambda_0^4} \leq \left\{ h(t)\,, \int_{M_t} H^2\, d\mu\,, \int_{M_t} |\nabla^m A|^2\, d\mu \right\} \leq \Lambda_0^4 \tag{4.11}$$

and

$$\max_{M_t} \left(|\mathring{A}| + |\nabla H| + |1 - hH| \right) \leq C^* \epsilon^{\frac{\alpha^2}{2}} e^{-\alpha\delta t} \leq C^* \epsilon^{\frac{\alpha^2}{2}}\,. \tag{4.12}$$

Here $0 < \alpha < 1$ is the universal constant from the Theorem 3.2 *and $\delta = \frac{1}{16n\Lambda_0^8|M_0|} > 0$. Then there exists some $\epsilon_3 = \epsilon_3(n, |M_0|, \Lambda_0, \alpha, C^*) > 0$ and $T_2 = T_2(\Lambda_0) > 0$ such that if*

$$\int_{M_0} |\mathring{A}|^2\, d\mu \leq \epsilon \leq \epsilon_3\,, \tag{4.13}$$

then (4.11) *and* (4.12) *hold for all $t \in [0, T + T_2]$.*

Proof. We start by applying the Theorem 3.2 while setting the initial time to be $t = T$. Then there exists $\epsilon_4 := \epsilon_0(n, |M_0|, \Lambda_0^4) > 0$ and $T_2 := T_1(\Lambda_0^4) > 0$ such that if

$$\int_{M_0} |\mathring{A}|^2\, d\mu \leq \epsilon \leq \epsilon_3\,,$$

then for all $t \in [T, T + T_2]$ we have

$$\max_{M_t} |A| \leq 2\Lambda_0^4 \quad \text{and} \quad \frac{1}{2\Lambda_0^4} \leq \left\{ h(t)\,, \int_{M_t} H^2\, d\mu \right\} \leq 2\Lambda_0^4, \tag{4.14}$$

and for some $\alpha \in (0,1)$,

$$\max_{M_t} \left(|\mathring{A}| + |\nabla H| + |1 - hH| \right) \leq C_1(n, |M_0|, \Lambda_0^4)\epsilon^{\alpha}\,. \tag{4.15}$$

Now we can choose $\epsilon_5 = \epsilon_5(n, |M_0|, \Lambda_0, \alpha) > 0$ sufficiently small such that for all $\epsilon < \epsilon_5$, we have

$$C_1(n, |M_0|, \Lambda_0^4)\epsilon^{\alpha - \frac{\alpha^2}{2}} \leq C^*$$

and therefore for all $t \in [0, T + T_2]$, we have

$$\max_{M_t} \left(|\mathring{A}| + |\nabla H| + |1 - hH| \right) \leq C^* \epsilon^{\frac{\alpha^2}{2}}\,. \tag{4.16}$$

We are now in position to apply the Theorem 3.6 as follows. Given (4.11) for $t \in [0,T]$, (4.14) for $t \in [T, T + T_2]$ and (4.16) for $t \in [0, T + T_2]$, we apply the Theorem 3.6 on the time interval $[0, T + T_2]$ with $\Lambda_1 = 2\Lambda_0^4, \hat{C} = C^*$ and $\beta = \frac{\alpha^2}{2}$ to find that there exists some $\epsilon_6 := \epsilon_1(n, |M_0|, \Lambda_0, C^*, \alpha) > 0$ sufficiently small, so that if $\epsilon \leq \epsilon_6$, then for all $t \in [0, T + T_2]$, we have

$$\begin{aligned} \max_{M_t} \left(|\mathring{A}| + |\nabla H| + |1 - hH| \right) &\leq C_2(n, |M_0|, 2\Lambda_0^4, C^*) \left(\max_{M_0} |\mathring{A}| \right)^{\alpha} e^{-\alpha\delta t} \\ &\leq C_2(n, |M_0|, 2\Lambda_0^4, C^*) \left(C_1(n, |M_0|, \Lambda_0) \right)^{\alpha} \epsilon^{\alpha^2} e^{-\alpha\delta t}\,, \end{aligned} \tag{4.17}$$

where $\delta = \frac{1}{16n\Lambda_0^8|M_0|} > 0$. Here we have also used the estimate (3.4) at $t = 0$.

We can then proceed to apply the Theorem 4.1. To do so, we first choose some $\epsilon_7 = \epsilon_7(n, |M_0|, \Lambda_0, \alpha, C^*) > 0$ small enough, so that

$$C_2(n, |M_0|, 2\Lambda_0^4, C^*) \left(C_1(n, |M_0|, \Lambda_0)\right)^\alpha \epsilon^{\frac{\alpha^2}{2}} \leq C^*. \tag{4.18}$$

This allows us to rewrite the estimate (4.17) to:

$$\max_{M_t} \left(|\mathring{A}| + |\nabla H| + |1 - hH|\right) \leq C^* \epsilon^{\frac{\alpha^2}{2}} e^{-\alpha\delta t}. \tag{4.19}$$

Comparing with (4.1) and (4.2), we can then apply the Theorem 4.1 to the time interval $[0, T + T_2]$ with $\Lambda_2 = 2\Lambda_0^4$, $\beta = \frac{\alpha^2}{2}$, $\delta = \frac{1}{16n\Lambda_0^8|M_0|}$, and $\tilde{C} = C^*$, so that we can choose

$$\epsilon_8 = \epsilon_8(n, |M_0|, \Lambda_0, \alpha, C^*) := \epsilon_2\left(n, |M_0|, 2\Lambda_0^4, C^*, \frac{\alpha^2}{2}, \frac{1}{16n\Lambda_0^8|M_0|}\right)$$

and if $\epsilon \leq \epsilon_8$ then we have (4.11).

We complete the proof by setting

$$\epsilon_3 = \epsilon_3(n, |M_0|, \Lambda_0, \alpha, C^*) = \min\{\epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8\} > 0\,.$$

$\square$

4.2. **Completion of the proof.** We now complete the proof of our main theorem:

Proof. (of Theorem 1.1) Suppose that the initial condition (3.1) is satisfied for some $\Lambda_0 \geq 100$ sufficiently large. Then by the Theorem 3.2, we first choose $\epsilon_0 = \epsilon_0(n, |M_0|, \Lambda_0) > 0$ and $T_1 = T_1(\Lambda_0) \in (0, 1]$, such that if $\epsilon \leq \epsilon_0$, then estimates (3.3) and (3.4) hold for all $t \in [0, T_1]$.

Now we can apply the Theorem 3.6 to the interval $[0, T_1]$ with $\Lambda_1 = 2\Lambda_0$, $\hat{C} = C_1(n, |M_0|, \Lambda_0)$, and $\beta = \alpha$, for some

$$\epsilon_9 = \epsilon_9(n, |M_0|, \Lambda_0, \alpha) := \epsilon_1\left(n, |M_0|, 2\Lambda_0, C_1(n, |M_0|, \Lambda_0), \alpha\right) > 0$$

sufficiently small such that if $\epsilon \leq \epsilon_9$, then for all $t \in [0, T_1]$, we have

$$\begin{aligned}
&\max_{M_t} \left(|\mathring{A}| + |\nabla H| + |1 - hH|\right) \\
&\leq C_2\left(n, |M_0|, 2\Lambda_0, C_1(n, |M_0|, \Lambda_0)\right) \left(\max_{M_0} |\mathring{A}|\right)^\alpha e^{-\alpha\gamma t} \\
&\leq C_2\left(n, |M_0|, 2\Lambda_0, C_1(n, |M_0|, \Lambda_0)\right) \left(C_1(n, |M_0|, \Lambda_0)\right)^\alpha \epsilon^{\frac{\alpha^2}{2}} e^{-\alpha\delta t}\,, \qquad (4.20)
\end{aligned}$$

where $\gamma = \frac{1}{16n\Lambda_0^2|M_0|} \geq \delta = \frac{1}{16n\Lambda_0^8|M_0|} > 0$, and we have again used the estimate (3.4) at $t = 0$.

Let $C^* = C_2\left(n, |M_0|, 2\Lambda_0, C_1(n, |M_0|, \Lambda_0)\right) \left(C_1(n, |M_0|, \Lambda_0)\right)^\alpha$, the above inequality (4.20) becomes

$$\max_{M_t} \left(|\mathring{A}| + |\nabla H| + |1 - hH|\right) \leq C^* \epsilon^{\frac{\alpha^2}{2}} e^{-\alpha\delta t}. \tag{4.21}$$

This allows us to apply the Theorem 4.2. We see that if we choose (note that $0 < \alpha < 1$ is some universal constant)

$$\epsilon \leq \epsilon_{10} = \epsilon_{10}(n, |M_0|, \Lambda_0) := \min\{\epsilon_9, \epsilon_3(n, |M_0|, \Lambda_0, \alpha, C^*)\}$$

then the flow (1.1) exists for all time and converges exponentially to a round sphere. □

References

[Aub98] Thierry Aubin, *Some nonlinear problems in Riemannian geometry*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.

[Che06] Xiuxiong Chen, *On the lower bound of energy functional E_1. I. A stability theorem on the Kähler Ricci flow*, J. Geom. Anal. **16** (2006), no. 1, 23–38.

[CLN06] Bennett Chow, Peng Lu, and Lei Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics, vol. 77, American Mathematical Society, Providence, RI, 2006.

[CLW09] Xiuxiong Chen, Haozhao Li, and Bing Wang, *Kähler-Ricci flow with small initial energy*, Geom. Funct. Anal. **18** (2009), no. 5, 1525–1563.

[ES98] Joachim Escher and Gieri Simonett, *The volume preserving mean curvature flow near spheres*, Proc. Amer. Math. Soc. **126** (1998), no. 9, 2789–2796.

[Ham82] Richard S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306.

[HS99] Gerhard Huisken and Carlo Sinestrari, *Mean curvature flow singularities for mean convex surfaces*, Calc. Var. Partial Differential Equations **8** (1999), no. 1, 1–14.

[Hui84] Gerhard Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. **20** (1984), no. 1, 237–266.

[Hui87] ———, *The volume preserving mean curvature flow*, J. Reine Angew. Math. **382** (1987), 35–48.

[Li09] Haozhao Li, *The volume-preserving mean curvature flow in Euclidean space*, Pacific J. Math. **243** (2009), no. 2, 331–355.

[McC03] James McCoy, *The surface area preserving mean curvature flow*, Asian J. Math. **7** (2003), no. 1, 7–30.

[McC04] ———, *The mixed volume preserving mean curvature flow*, Math. Z. **246** (2004), no. 1-2, 155–166.

[Pih98] D. Pihan, *A length preserving geometric heat flow for curves*, thesis, Univ. of Melbourne (1998).

[Top08] Peter Topping, *Relating diameter and mean curvature for submanifolds of Euclidean space*, Comment. Math. Helv. **83** (2008), no. 3, 539–546.

[Ye93] Rugang Ye, *Ricci flow, Einstein metrics and space forms*, Trans. Amer. Math. Soc. **338:2** (1993), 871–896.

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